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LETTERS TO THE EDITOR

THE MOTION OF A TWO-MASS SYSTEM WITH NON-LINEAR CONNECTION

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1. INTRODUCTION

The dynamics of a two-mass system connected with a stiffness has been widely analyzed [1, 2]. Usually, it is assumed that the property of the stiffness is linear. For that case it is evident that the motion is a combination of a translation and a vibration with only one eigenfrequency in spite of the fact that the system has two degrees of freedom. In this paper the problem of a two-mass system is extended: it is assumed that the stiffness is non-linear. The non-linearity is of a cubic type. The mathematical model of the system contains two ordinary conjugate differential equations with cubic non-linearities.

Recently, some researchers have presented several techniques for solving analytically a second order differential equation with various strong non-linearities. For instance, Yuste and Bejarano [3] developed an elliptic Krylov–Bogolubov method with Jacobian elliptic functions for solving a differential equation with cubic non-linearity. Coppola and Rand [4] used symbolic computation to implement an averaging method with elliptic functions. An averaging method using generalized harmonic functions for strongly non-linear oscillators is presented in the paper of Xu and Cheung [5]. Belhaq and Lakrad [6] adopted the multiple scales method for a class of autonomous strongly non-linear oscillators. The elliptic-harmonic-balance method is used to study mixed parity non-linear oscillators [7]. In reference [8] the exact solutions for the hard singular non-linear oscillators are determined. All the methods mentioned above have their own advantages for obtaining approximate analytical solutions for a one-degree-of-freedom oscillator.

Using the previous-mentioned results some particular solutions of the differential equation with complex functions describing the vibrations of a rotor which rotates with a constant angular velocity are determined [9]. The mentioned differential equation is obtained by introducing a complex deflection function in the system of two differential equations. The strong non-linear vibrations of the rotors with variable mass are described with a differential equation with complex functions and analytically solved in reference [10]. The harmonic balance method, the elliptic-Krylov–Bogolubov method and the elliptic perturbation method [11] are extended for obtaining the approximate solutions of the differential equation with complex functions and a strong cubic non-linearity [12].

Based on the previous studies, the general analytical solution of a special type of two coupled differential equations with strong cubic non-linearity is determined in this paper in the closed form.

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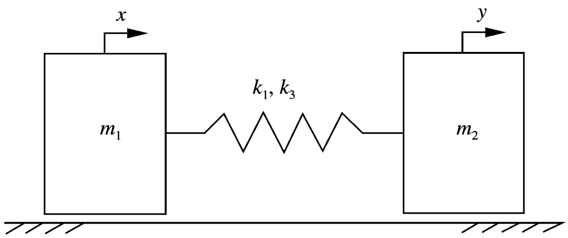

Figure 1. Model of the two-mass system with stiffness.

2. ANALYTICAL SOLUTION

The model of the system is shown in Figure 1. Two masses, m_1 and m_2, are connected with a stiffness whose linear coefficient of rigidity is k_1 and the non-linear coefficient is k_3. The system has two degrees of freedom. The generalized co-ordinates of the system are x and y. The motion of the system is described by

$$m_1\ddot{x} + k_1(x-y) + k_3(x-y)^3 = 0,$$
$$m_2\ddot{y} + k_1(y-x) + k_3(y-x)^3 = 0, \tag{1}$$

where $(\cdot\cdot) \equiv d^2/dt^2$. The differential equations (1) represent a system of two coupled non-linear differential equations with strong cubic non-linearities. The system of equations (1) is now solved.

To simplify these equations introduce a new variable

$$X = y - x. \tag{2}$$

The system of equations (1) transforms to

$$m_1\ddot{x} - k_1X - k_3X^3 = 0,$$
$$m_2(\ddot{x} + \ddot{X}) + k_1X + k_3X^3 = 0. \tag{3}$$

Eliminating the variable x a single second order non-linear differential equation is obtained:

$$\mu\ddot{X} + k_1X + k_3X^3 = 0, \tag{4}$$

where the reduced mass μ is given by

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}. \tag{5}$$

The differential equation (4) represents a differential equation with strong cubic non-linearity and its solution has the form of Jacobi elliptic functions (see references [3, 4]. The closed-form analytical solution is

$$X = A\,\mathrm{cn}(\omega t + \theta, k^2), \tag{6}$$

where $\mathrm{cn}(\omega t + \theta, k^2)$ is a Jacobian elliptic function [13], with the frequency parameter ω:

$$\omega^2 = \frac{1}{\mu}(k_1 + A^2k_3) \tag{7}$$

and the modulus k^2;

$$k^2 = \frac{k_3A^2}{2(k_1 + k_3A^2)}. \tag{8}$$

It is worth saying that the frequency of the function depends on the mass ratio λ. The modulus of the elliptic function does not depend on the mass ratio. A and θ are the unknown coefficients.

Summarizing equations (3)

$$\ddot{x} = -\frac{1}{\lambda}\ddot{X}, \tag{9}$$

where

$$\frac{1}{\lambda} = \frac{\mu}{m_1}.$$

Two integrations of equation (9) gives

$$x = -\frac{1}{\lambda}X + Bt + C, \tag{10}$$

where B and C are constants of integrations.

Substituting equation (10) into equation (2) gives

$$y = X\left(1 - \frac{1}{\lambda}\right) + Bt + C. \tag{11}$$

Using solution (6) with equations (7) and (8) the general solution of the system of equation (1) is obtained as

$$x = -\frac{A}{\lambda}\,\mathrm{cn}\,(\omega t + \theta, k^2) + Bt + C,$$

$$y = \left(1 - \frac{1}{\lambda}\right) A\ \mathrm{cn}\,(\omega t + \theta, k^2) + Bt + C, \tag{12}$$

where A, θ, B and C are the unknown coefficients which depend on the initial conditions.

3. CONNECTION BETWEEN THE INITIAL CONDITIONS AND THE PARAMETERS OF THE SOLUTION

Now consider the initial conditions in general form:

$$x(0) = x_0, \qquad y(0) = y_0, \qquad \dot{x}(0) = \dot{x}_0, \qquad \dot{y}(0) = \dot{y}_0, \tag{13}$$

where $(\dot{\ }) \equiv \mathrm{d}/\mathrm{d}t$. Substituting equation (13) into equation (12) one obtains the following four equations:

$$x_0 = -\frac{A}{\lambda}\,\mathrm{cn}\,(\theta, k^2) + C,$$

$$y_0 = \left(1 - \frac{1}{\lambda}\right) A\,\mathrm{cn}\,(\theta, k^2) + C,$$

$$\dot{x}_0 = \frac{A\omega}{\lambda}\,\mathrm{sn}\,(\theta, k^2)\,\mathrm{dn}\,(\theta, k^2) + B,$$

$$\dot{y}_0 = -\left(1 - \frac{1}{\lambda}\right) A\omega\,\mathrm{sn}(\theta, k^2)\,\mathrm{dn}\,(\theta, k^2) + B, \tag{14}$$

where sn and dn are Jacobi elliptic functions [5].

Solving equations (14) gives

$$B = \frac{1}{\lambda}\,[\dot{y}_0 + (\lambda - 1)\,\dot{x}_0],$$

$$C = \left(1 - \frac{1}{\lambda}\right) x_0 + \frac{1}{\lambda}\,y_0,$$

$$A = \pm\sqrt{\frac{(y_0 - x_0)^2 + (\dot{y}_0 - \dot{x}_0)^{2m_1/\lambda}}{k_1 + k_3\,(y_0 - x_0)^2}}, \tag{15}$$

and θ is the solution of the equation

$$\mathrm{sc}\,(\theta, k^2)\,\mathrm{dn}\,(\theta, k^2) = \frac{1\,(\dot{y}_0 - \dot{x}_0)}{\omega\,(y_0 - x_0)}, \tag{16}$$

where

$$\omega^2 = \frac{1}{\mu}\,[k_1 + k_3(y_0 - x_0)^2] + \frac{k_3\,(\dot{y}_0 - \dot{x}_0)^2}{k_1 + k_3\,(y_0 - x_0)^2}, \tag{17}$$

$$k^2 = \frac{k_3\,(y_0 - x_0)^2\,[k_1 + k_3\,(y_0 - x_0)^2] + \mu k_3\,(\dot{y}_0 - \dot{x}_0)^2}{2\,[k_1 + k_3\,(y_0 - x_0)^2]^2 + 2k_3\mu(\dot{y}_0 - \dot{x}_0)^2}. \tag{18}$$

The function sc = sn/cn is a Jacobi elliptic function [13]. Substituting equations (15)–(18) into equation (12) the closed-form solution of system (1) is obtained. Analyzing the obtained solutions it can be concluded that the motion of the masses contains the translation with the constant velocity B and an oscillatory motion with a frequency ω and period $4K\,(k^2)$, where $K\,(k^2)$ is the complete first order elliptic integral. This is shown by means of an example.

Example 1. Consider a two equal unit mass system ($m_1 = m_2 = 1$) connected by a spring with non-linear properties. The coefficients of rigidity are $k_1 = 1$ dyne/cm, $k_3 = 1$ dyne/cm^3. For the initial conditions (in cm)

$$x_0 = 0{\cdot}5, \qquad \dot{x}_0 = 0{\cdot}5, \qquad y_0 = 0{\cdot}1, \qquad \dot{y}_0 = 0{\cdot}1. \tag{19}$$

and according to equation (12) the solution of equations (1) is

$$x = -\,0{\cdot}22743\,\mathrm{cn}(1{\cdot}5678t + 2{\cdot}73, \quad 0{\cdot}1071) + 0{\cdot}3t + 0{\cdot}3,$$

$$y = 0{\cdot}22743\,\mathrm{cn}(1{\cdot}5678t + 2{\cdot}73, \quad 0{\cdot}1071) + 0{\cdot}3t + 0{\cdot}3. \tag{20}$$

The solutions (20) are plotted in Figure 2. It can be seen that the position of the masses depends on the initial position ($C = 0{\cdot}3$ cm), initial velocity of translation ($B = 0{\cdot}3$ cm/s) and the amplitude of vibrations ($A_1 = 0{\cdot}22743$ cm), its frequency ($\omega = 1{\cdot}5678\,\mathrm{s}^{-1}$) and period of vibration ($4K$ $(0{\cdot}1071) = 6{\cdot}4622$).

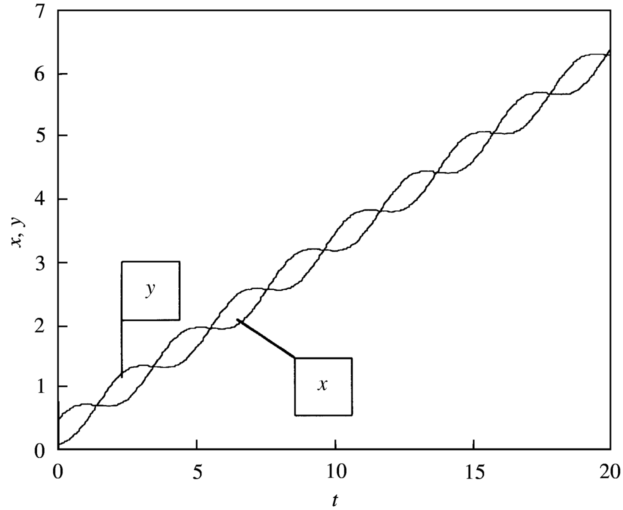

Figure 2. x–t and y–t diagrams for the following initial conditions in cm: $x(0) = 0{\cdot}5$, $x(0) = 0{\cdot}5$, $y(0) = 0{\cdot}1$, $y(0) = 0{\cdot}1$.

3.1. SPECIAL INITIAL CONDITIONS

Very often the motion starts with zero velocity i.e.,

$$x(0) = x_0, \qquad y(0) = y_0, \qquad \dot{x}_0 = \dot{y}_0 = 0. \tag{21}$$

Then the coefficients which depend on the initial conditions are

$$A = y_0 - x_0, \qquad B = 0, \qquad C = \left(1 - \frac{1}{\lambda}\right)x_0 + \frac{y_0}{\lambda}, \qquad \theta = 0, \tag{22}$$

and the solutions of equation (1) are

$$x = \frac{x_0 - y_0}{\lambda} \operatorname{cn}(\omega t, k^2) + \left(1 - \frac{1}{\lambda}\right)x_0 + \frac{y_0}{\lambda},$$

$$y = \left(1 - \frac{1}{\lambda}\right)(y_0 - x_0) \operatorname{cn}(\omega t, k^2) + \left(1 - \frac{1}{\lambda}\right)x_0 + \frac{y_0}{\lambda}, \tag{23}$$

where

$$\omega^2 = \frac{1}{\mu}\,[k_1 + k_3\,(y_0 - x_0)^2], \tag{24}$$

$$k^2 = \frac{k_3(y_0 - x_0)^2}{2[k_1 + k_3(y_0 - x_0)^2]}. \tag{25}$$

It means that the motion of both the masses is oscillatory. The masses vibrate around a fixed position determined by the initial conditions, i.e.,

$$x_s = y_s = \left(1 - \frac{1}{\lambda}\right)x_0 + \frac{y_0}{\lambda}. \tag{26}$$

To illustrate the statement consider an example.

Example 2. Consider the same model as in the previous example but with the following initial conditions (in cm):

$$x_0 = 0, \qquad \dot{x}_0 = 0, \qquad y_0 = 1, \qquad \dot{y}_0 = 0. \tag{27}$$

The motion of both the masses are according to equation (23)

$$x = \tfrac{1}{2}\,[1 - \mathrm{cn}(2t, \tfrac{1}{4}],$$
$$y = \tfrac{1}{2}\,[1 + \mathrm{cn}(2t, \tfrac{1}{4}]. \tag{28}$$

The functions (28) are plotted in Figure 3. Both masses in the system have "harmonic" oscillatory motions with amplitude $A_1 = 0{\cdot}5$ cm, frequency $\omega = 2\,\mathrm{s}^{-1}$ and period of vibrations $4K(0{\cdot}25) = 6{\cdot}743$ around the fixed position ($x_s = y_s = 0{\cdot}5$ cm).

4. A STIFFNESS WITH PURE-CUBIC RIGIDITY

Consider the case when the spring which connects the masses has a rigidity which is a pure non-linear function of deflection of the masses. Then $k_1 = 0$ and the coefficient of the cubic non-linearity is k_3. The differential equations of motion are

$$m_1\,\ddot{x} + k_3\,(x - y)^3 = 0,$$
$$m_2\,\ddot{y} + k_3(y - x)^3 = 0. \tag{29}$$

Using the suggested procedure the solution of equation (29) is a special case of equation (12) and it is

$$x = -\frac{A}{\lambda}\,\mathrm{cn}\,(\omega t + \theta,\ 1/2) + Bt + C,$$

$$y = \left(1 - \frac{1}{\lambda}\right) A\,\mathrm{cn}\,(\omega t + \theta,\ 1/2) + Bt + C, \tag{30}$$

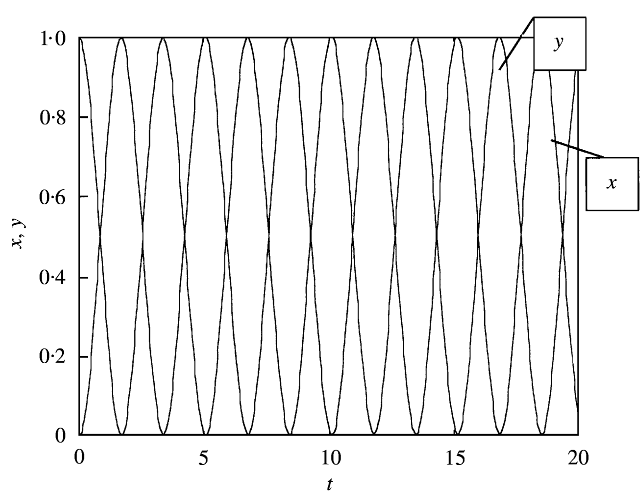

Figure 3. x–t and y–t diagrams for the following initial conditions in cm: $x(0) = 0$, $x(0) = 0$, $y(0) = 1$, $y(0) = 0$.

where the parameters of the elliptic function (7) and (8) are

$$k^2 = \frac{1}{2}, \qquad \omega^2 = \frac{A^2 k_3}{\mu}. \tag{31}$$

The modulus of the elliptic function has a constant value. The period of vibrations is constant and it is

$$4K(1/2) = 7{\cdot}4163 \tag{32}$$

where $K(1/2)$ is the complete elliptic integral of the first kind [6].

5. A LINEAR SPRING

Assume that the spring which connects the masses is a linear one.

$$m_1 \ddot{x} + k_1 (x - y) = 0,$$
$$m_2 \ddot{y} + k_1 (y - x) = 0. \tag{33}$$

For $k_3 = 0$ the parameters in relations (7) and (8) are

$$\omega^2 = \frac{k_1}{\mu}, \qquad k^2 = 0. \tag{34}$$

The Jacobian elliptic function transforms to a harmonic function (see reference [14])

$$\mathrm{cn}\,(\omega t + \theta, 0) \equiv \cos(\omega t + \theta).$$

The modified solutions (12) of the linear system are

$$x = -\frac{A}{\lambda}\cos(\omega t + \theta) + Bt + C,$$
$$y = \left(1 - \frac{1}{\lambda}\right) A \cos(\omega t + \theta) + Bt + C \tag{35}$$

and are well known in the literature.

6. A SPRING WITH SOFT NON-LINEARITY

For the case when the non-linearity is soft the constant k_3 is negative and the differential equations of motion are

$$m_1 \ddot{x} + k_1(x - y) - k_3(x - y)^3 = 0,$$
$$m_2 \ddot{y} + k_1(y - x) - k_3(y - x)^3 = 0. \tag{36}$$

The general solution of the system is according to equation (12)

$$x = -\frac{A}{\lambda}\,\mathrm{cn}\,(\omega^* t + \theta, k^{*2}) + Bt + C,$$
$$y = \left(1 - \frac{1}{\lambda}\right) A\,\mathrm{cn}\,(\omega^* t + \theta, k^{*2}) + Bt + C, \tag{37}$$

where the coefficients of the Jacobi elliptic function are

$$\omega^{*2} = \frac{1}{\mu}(k_1 - A^2 k_3), \qquad k^{*2} = -\frac{k_3 A^2}{2(k_1 - k_3 A^2)}.$$

The solution is valid only for

$$k_1 - A^2 k_3 > 0.$$

It means that the modulus of the elliptic function is negative. Using the transformation of the elliptic function with a negative modulus to an elliptic function with a positive modulus (see reference [14]) the general solution of the system (36) as

$$x = -\frac{A}{\lambda}\,\mathrm{cd}\,[p(\omega^* t + \theta), k^{**2}] + Bt + C,$$

$$y = \left(1 - \frac{1}{\lambda}\right) A\,\mathrm{cd}\,[p(\omega^* t + \theta), k^{**2}] + Bt + C, \tag{38}$$

where

$$k^{**2} = \frac{k_3 A^2}{2k_1 - k_3 A^2}, \qquad p = \frac{2(k_1 - k_3 A^2)}{2k_1 - k_3 A^2} \tag{39}$$

and cd is a periodical Jacobi elliptic function [13] with period $4K\ (k^{**2})$. The motion has the same properties as for the case of hard non-linearity ($k_3 > 0$) which is discussed in this paper.

7. CONCLUSIONS

It can be concluded that the motion of a two-mass system connected with a cubic non-linear stiffness, described with a system of two second order ordinary coupled differential equations with strong cubic non-linearities, can be described analytically. There exist the exact general solutions of this special type of differential equations in closed form. It is evident that the motion of the masses is a combination of a translation and an oscillatory motion. The type of motion is the same as for the case of linear stiffness. The differences between linear and non-linear connections are the following:

1. For the linear case the period of vibrations is constant. For the non-linear case it depends on the coefficient of non-linearity and the amplitude of vibrations i.e., on the initial conditions.
2. The frequency of vibrations is constant for the linear case and depends on the initial conditions for the non-linear case.

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